Calculus 1

Final Exam – Solutions

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1) Apply L'Hospital's Rule to evaluate the limit $\lim_{x\to a} \frac{a^x - x^a}{x-a}$, where $a > 0$. Indicate the results (e.g. limit laws, continuity, differentiation rules) used in each step.

Solution. The limit is an indeterminate form of type "$0/0$". Thus we can directly apply l'Hospital's Rule. We start by expressing the numerator as follows

$$a^x - x^a = e^{x\ln a} - x^a,$$

which is valid as $a > 0$. Let us now use l'Hospital's Rule:

$$\lim_{x\to a} \frac{e^{x\ln a} - x^a}{x-a} \overset{\text{l'H}}{=} \lim_{x\to a} \frac{e^{x\ln a}\ln a - a x^{a-1}}{1} = \lim_{x\to a}(a^x \ln a - a x^{a-1}) = a^a(\ln a - 1).$$

Above we used the Chain Rule, the Difference Rule, the Basic Derivatives $(e^x)' = e^x$, $(x)' = 1$, $(c)' = 0$, and the Generalized Power Rule $(x^a)' = a x^{a-1}$. The last equality follows from the continuity of the exponential and power functions and the Difference Law of Limits. Therefore we have found that

$$\lim_{x\to a} \frac{a^x - x^a}{x-a} = a^a(\ln a - 1).$$

2) Use Taylor Series to find the limit $\lim_{x\to\infty} \left(\sqrt[6]{x^6+x^5} - \sqrt[6]{x^6-x^5}\right)$.

Solution. Recall that according to the Binomial Theorem, we have

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} z^k, \quad |z| < 1.$$

Therefore the terms in the limit can be expressed as follows

$$\sqrt[6]{x^6 \pm x^5} = x\sqrt[6]{1 \pm x^{-1}} = x(1 \pm x^{-1})^{1/6} = x \sum_{k=0}^{\infty} \binom{1/6}{k} (\pm x)^{-k}$$

if $|\pm x^{-1}| = |x^{-1}| < 1$ or equivalently $|x| > 1$ holds. This is so as we take $x \to \infty$. Due to absolute convergence (guaranteed by the Ratio Test) the series may be multiplied by x and subtracted from one another term-by-term. This results in the following series

$$\sqrt[6]{x^6+x^5} - \sqrt[6]{x^6-x^5} = \sum_{k=0}^{\infty} \binom{1/6}{k} [1 - (-1)^k] x^{1-k}.$$

Since $[1-(-1)^k] = 0$ if $k = 2m$ is even and $[1-(-1)^k] = 2$ if $k = 2m+1$ is odd, the series contains only the even powers of $1/x$, that is

$$\sqrt[6]{x^6+x^5} - \sqrt[6]{x^6-x^5} = \sum_{m=0}^{\infty} \binom{1/6}{2m+1} 2x^{1-(2m+1)} = 2\sum_{m=0}^{\infty} \binom{1/6}{2m+1} x^{-2m}.$$

Note that the constant term ($m = 0$) reads

$$2\binom{1/6}{1} = 2 \cdot \frac{1}{6} = \frac{1}{3}$$

and the series can be written as follows

$$\sqrt[6]{x^6 + x^5} - \sqrt[6]{x^6 - x^5} = \frac{1}{3} + \frac{2}{x^2} \sum_{m=1}^{\infty} \binom{1/6}{2m+1} x^{-2m+2} = \frac{1}{3} + \frac{2}{x^2} \sum_{\ell=0}^{\infty} \binom{1/6}{2\ell+3} x^{-2\ell}$$

By introducing $t = x^{-2}$ the non-constant remainder may be shown to tend to zero as we have

$$\frac{2}{x^2} \sum_{\ell=0}^{\infty} \binom{1/6}{2\ell+3} x^{-2\ell} = 2t \sum_{\ell=0}^{\infty} \binom{1/6}{2\ell+3} t^{\ell}$$

which in absolute value, due to $t > 0$ and

$$\left| \binom{1/6}{2\ell+3} \right| = \left| \frac{\frac{1}{6}\left(\frac{1}{6} - 1\right) \ldots \left(\frac{1}{6} - (2\ell+3) + 1\right)}{1 \cdot 2 \cdot \ldots \cdot (2\ell+3)} \right| < 1 \quad \Leftarrow \quad \left| \frac{\frac{1}{6} - j + 1}{j} \right| < 1 \quad (j = 1, 2, 3, \ldots)$$

acquires the following bounds

$$0 < \left| 2t \sum_{\ell=0}^{\infty} \binom{1/6}{2\ell+3} t^{\ell} \right| \le 2t \sum_{\ell=0}^{\infty} \left| \binom{1/6}{2\ell+3} \right| t^{\ell} < 2t \sum_{\ell=0}^{\infty} t^{\ell} = \frac{2t}{1-t}$$

where we used the Triangle Inequality and the Geometric Series Formula. Finally, as $x \to \infty$, $t = x^{-2} \to 0^+$ and therefore by the Squeeze Theorem

$$\lim_{t \to 0^+} 2t \sum_{\ell=0}^{\infty} \binom{1/6}{2\ell+3} t^{\ell} = 0.$$

In conclusion, we have shown that $\lim\limits_{x \to \infty} \left(\sqrt[6]{x^6 + x^5} - \sqrt[6]{x^6 - x^5} \right) = 1/3$.

3) Calculate the arc length of the curve $y = \ln(\sin x)$, $\quad \frac{\pi}{6} \le x \le \frac{\pi}{2}$.

Solution. The curve can be viewed as the graph of the function $f(x) = \ln(\sin x)$ with $\frac{\pi}{6} \le x \le \frac{\pi}{2}$. Thus its arc length L is given by the following integral

$$L = \int_{\pi/6}^{\pi/2} \sqrt{1 + [f'(x)]^2}\, dx.$$

Since the derivative of $f(x) = \ln\big(\sin(x)\big)$ is

$$f'(x) = [\ln(\sin x)]' = \frac{1}{\sin x}[\sin x]' = \frac{\cos x}{\sin x},$$

where we used the Chain Rule and a Basic (Trigonometric) Derivative, the integrand is found to be

$$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + \frac{\cos^2 x}{\sin^2 x}} = \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} = \sqrt{\frac{1}{\sin^2 x}} = \frac{1}{\sin x}.$$

Here we used the trigonometric identity $\sin^2 x + \cos^2 x = 1$ and that $\sin x > 0$ if $x \in [\pi/6, \pi/2]$. Hence the arc length can be found via Weierstrass' magic substitution $u = \tan\frac{x}{2}$, which yields $\sin x = \frac{2u}{1+u^2}$ and $dx = \frac{2}{1+u^2}\,du$ and

$$L = \int_{\pi/6}^{\pi/2} \frac{1}{\sin x}\,dx = \int_{\tan(\pi/12)}^{\tan(\pi/4)} \frac{1+u^2}{2u}\frac{2}{1+u^2}\,du = \int_{\tan(\pi/12)}^{\tan(\pi/4)} \frac{1}{u}\,du$$

$$= [\ln u]_{u=\tan(\pi/2)}^{u=\tan(\pi/4)} = \ln\big(\tan(\pi/4)\big) - \ln\big(\tan(\pi/12)\big)$$

$$= \ln(1) - \ln\big(\tan(\pi/12)\big) = -\ln\big(\tan(\pi/12)\big) \approx 1.317.$$

Here we used that $\tan(\pi/4) = 1$. The arc length is $L = -\ln\big(\tan(\pi/12)\big) \approx 1.317$

Remark. We may simplify the answer further by applying the tangent half-angle identity $\tan(x/2) = (1-\cos x)/\sin x$ with $x = \pi/6$ to find that $\tan(\pi/12) = 2 - \sqrt{3}$ and obtain

$$L = -\ln\big(2-\sqrt{3}\big) = \ln\left(\frac{1}{2-\sqrt{3}}\right) = \ln\left(\frac{1}{2-\sqrt{3}}\cdot\frac{2+\sqrt{3}}{2+\sqrt{3}}\right) = \ln(2+\sqrt{3}).$$

4) Evaluate the integral $\int_0^\infty e^{-Ax}\cos x\,dx$ in terms of the constant $A > 0$.

Solution. Method #1 (Integration by Parts × 2). We encountered this type of integral before (as an indefinite integral) and managed to solve it using integration by parts twice in a row. If we choose $u = e^{-Ax}$ and $dv = \cos x\,dx$, then $du = -Ae^{-Ax}\,dx$ and $v = \sin x$ and therefore

$$\int_0^\infty e^{-Ax}\cos x\,dx = \lim_{b\to\infty}\int_0^b e^{-Ax}\cos x\,dx = \lim_{b\to\infty}[e^{-Ax}\sin x]_0^b + A\lim_{b\to\infty}\int_0^b e^{-Ax}\sin x\,dx.$$

The first term on the right-hand side vanishes as $\sin 0 = 0$ and $0 \le |\sin x|e^{-Ax} \le e^{-Ax} \to 0$ as $x \to \infty$ (cf. Squeeze Theorem). Thus we have

$$\int_0^\infty e^{-Ax}\cos x\,dx = A\int_0^\infty e^{-Ax}\sin x\,dx.$$

Integrating by parts again and choosing $u = e^{-Ax}$ and $dv = \sin x\,dx$ this time, we get $du = -Ae^{-Ax}\,dx$ and $v = -\cos x$ and therefore

$$\int_0^\infty e^{-Ax}\sin x\,dx = \lim_{b\to\infty}\int_0^b e^{-Ax}\sin x\,dx = \lim_{b\to\infty}[-e^{-Ax}\cos x]_0^b - A\lim_{b\to\infty}\int_0^b e^{-Ax}\cos x\,dx.$$

The first term on the right-hand side is $-(-e^0\cos 0) = 1$ since $0 \le |\cos x|e^{-Ax} \le e^{-Ax} \to 0$ as $x \to \infty$ (Squeeze Theorem). Therefore we obtain

$$\int_0^\infty e^{-Ax}\cos x\,dx = A\int_0^\infty e^{-Ax}\sin x\,dx = A\left(1 - A\int_0^\infty e^{-Ax}\cos x\,dx\right).$$

Solving this equation for the integral in question, we find that

$$\int_0^\infty e^{-Ax}\cos x\,dx = \frac{A}{A^2+1}.$$

Method #2 (Using Euler's formula). Expressing the cosine function in terms of complex exponentials via Euler's formula yields

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}.$$

Thus the integral in question can be written as

$$\int_0^\infty e^{-Ax}\cos x\,dx = \frac{1}{2}\int_0^\infty (e^{(-A+i)x} + e^{(-A-i)x})\,dx$$

and we find that[1]

$$\int_0^\infty e^{-Ax}\cos x\,dx = \frac{1}{2}\lim_{b\to\infty}\left[\frac{e^{(-A+i)x}}{-A+i} + \frac{e^{(-A-i)x}}{-A-i}\right]_0^b$$

The right-hand side simply evaluates to

$$\frac{1}{2}\left(\frac{1}{i-A} - \frac{1}{i+A}\right) = \frac{1}{2}\frac{2A}{A^2+1} = \frac{A}{A^2+1}$$

since $0 < |e^{(-A\pm i)x}| = e^{-Ax} \to 0$ as $x \to \infty$ (Squeeze Theorem). In conclusion, we have

$$\int_0^\infty e^{-Ax}\cos x\,dx = \frac{A}{A^2+1}.$$

5) Solve the initial value problem $y'(x) + x\,y(x) = x$, $y(0) = 2$.

Solution. Method #1 (Separation of variables). By rearranging the equation as $y' = (1-y)x$, we notice that it is a separable ODE. Dividing both sides by $y-1$ yields

$$\frac{y'}{y-1} = -x.$$

Note that the division by $y-1$ implicitly ignores the constant function $y(x) = 1$, which is a valid solution of the ODE $y' + x\,y = x$. However, this does not cause a problem as the initial value $y(0) = 2 \neq 1$ discards this solution. By integrating both sides with respect to x, we get

$$\int \frac{y'}{y-1}\,dx = \int (-x)\,dx$$

which evaluates to

$$\ln|y-1| = -\frac{x^2}{2} + C \quad \Rightarrow \quad y(x) = 1 \pm e^C\,e^{-x^2/2}.$$

Therefore the general solution reads

$$y(x) = 1 + Ae^{-x^2/2},$$

where A is an arbitrary constant. To find the particular solution, we employ the initial value

$$2 = y(0) = 1 + Ae^{-0^2/2} = 1 + A(1) = 1 + A \quad \Rightarrow \quad A = 1.$$

In summary, the solution of the initial value problem $y'(x) + x\,y(x) = x$, $y(0) = 2$ is

$$y(x) = 1 + e^{-x^2/2}.$$

[1] There's some subtlety here as we first need to show that $\frac{d}{dx}(e^{(a+bi)x}) = (a+bi)e^{(a+bi)x}$ which follows from a term-by-term differentiation of the Taylor series (permitted by absolute convergence).

Method #2 (Integrating factor). The ODE, being of the form $y' + P(x)y = Q(x)$, is a linear first-order equation, therefore we may use the integrating factor method. The integrating factor reads

$$I(x) = e^{\int P(x)\,dx} = e^{\int x\,dx} = e^{x^2/2}.$$

Multiplying both sides of the ODE by I turns the equation into

$$y'e^{-x/2} + y(xe^{x^2/2}) = xe^{x^2/2}.$$

Note that the left-hand side is the derivative of $ye^{-x^2/2}$ (by the Product Rule), i.e.

$$(ye^{-x/2})' = xe^{x^2/2}.$$

Integration on both sides with respect to x leads to the following equation

$$ye^{-x/2} = \int xe^{x^2/2}\,dx.$$

By performing the integral on the right-hand side (e.g. via substituting $u = x^2/2$), we find that

$$ye^{-x/2} = e^{x^2/2} + A$$

and therefore the general solution is found

$$y(x) = 1 + Ae^{-x^2/2},$$

where A is an arbitrary constant. To find the particular solution, we employ the initial value

$$2 = y(0) = 1 + Ae^{-0^2/2} = 1 + A(1) = 1 + A \quad \Rightarrow \quad A = 1.$$

In summary, the solution of the initial value problem $y'(x) + x\,y(x) = x$, $y(0) = 2$ is

$$y(x) = 1 + e^{-x^2/2}.$$

Method #3 (Variation of constant). The ODE, being of the form $y' + P(x)y = Q(x)$, is a linear first-order equation, therefore we may use the variation of constant method. We first solve the homogeneous equation (which is a separable ODE)

$$y_h' + xy_h = 0,$$

We find the general solution of the homogeneous equation to be

$$y_h(x) = e^{\int(-x)\,dx} = e^{-x^2/2+C} = Ae^{-x^2/2}.$$

The solution of the non-homogeneous equation is obtained by varying the parameter A, that is looking for y in the form

$$y(x) = A(x)e^{-x^2/2}.$$

Plugging this back into the original ODE, we find that

$$A'(x)e^{-x^2/2} = x \quad \Rightarrow \quad A'(x) = xe^{x^2/2} \quad \Rightarrow \quad A(x) = \int xe^{x^2/2}\,dx = e^{x^2/2} + \tilde{C}.$$

Therefore the general solution reads

$$y(x) = 1 + \tilde{C}e^{-x^2/2},$$

where A is an arbitrary constant. To find the particular solution, we employ the initial value

$$2 = y(0) = 1 + \tilde{C}e^{-0^2/2} = 1 + \tilde{C}(1) = 1 + \tilde{C} \quad \Rightarrow \quad \tilde{C} = 1.$$

In summary, the solution of the initial value problem $y'(x) + x\,y(x) = x$, $y(0) = 2$ is

$$y(x) = 1 + e^{-x^2/2}.$$

6) Solve the following initial value problem

$$y''(x) + 6y'(x) + 9y(x) = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution. The auxiliary equation for the above ODE can be written as

$$r^2 + 6r + 9 = 0.$$

This quadratic has repeated real roots, namely

$$r_{1,2} = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 9}}{2} = -3.$$

This means resonance, therefore the general solution of the differential equation takes the following form

$$y(x) = (c_1 + c_2 x)e^{-3x}.$$

To find the values of the constants c_1, c_2 we need to compute the derivative of $y(x)$:

$$y'(x) = [c_2 - 3(c_1 + c_2 x)]e^{-3x}$$

Therefore we have

$$1 = y(0) = c_1$$

and

$$1 = y'(0) = c_2 - 3c_1 = c_2 - 3$$

These imply that $c_1 = 1, c_2 = 4$. In conclusion, the solution of the initial value problem $y'' + 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = 1$ is

$$y(x) = (1 + 4x)e^{-3x}.$$